Lecture 36. Spectral theorem

Def A matrix A with $A^T = A$ is called a symmetric matrix.

e.g. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\begin{bmatrix} 8 & 2 & 1 \\ 2 & 3 & -4 \\ 1 & -4 & 5 \end{bmatrix}$

Note (1) A symmetric matrix must be a square matrix.

(its transpose must be of the same size)

(2) The entries of a symmetric matrix are symmetric with respective to the main diagonal.

Prop Given an arbitrary matrix A, both AA^T and A^TA are symmetric.

Note Such matrices arise from orthogonal projections, reflections, and least squares problems.

Thm (The spectral theorem)

Let A be a symmetric matrix.

(1) All eigenvalues of A are real numbers.

(2) Eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

(3) There exists an orthonormal eigenbasis for A.

Prop Given an $n \times n$ matrix A whose columns form an orthonormal basis of $\mathbb{R}^n$, we have $A^{-1} = A^T$.

pf Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ be the columns of A.

$\Rightarrow$ $\vec{v}_i \cdot \vec{v}_i = \|\vec{v}_i\|^2 = 1$ and $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$

A^T has rows $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$.

$$\Rightarrow A^T A = \begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \cdots & \vec{v}_1 \cdot \vec{v}_n \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_3 & \cdots & \vec{v}_2 \cdot \vec{v}_3 \\ \vdots & \vdots & & \vdots \\ \vec{v}_n \cdot \vec{v}_1 & \vec{v}_n \cdot \vec{v}_2 & \cdots & \vec{v}_n \cdot \vec{v}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

$\Rightarrow$ A^T is the inverse of A

e.g. $A = \frac{1}{7}\begin{bmatrix} 6 & -2 & 3 \\ -2 & 3 & 6 \\ -3 & -6 & 2 \end{bmatrix}$

$\Rightarrow$ A has orthonormal columns $\vec{v}_1 = \frac{1}{7}\begin{bmatrix} 6 \\ -2 \\ -3 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{7}\begin{bmatrix} -2 \\ 3 \\ -6 \end{bmatrix}$, $\vec{v}_3 = \frac{1}{7}\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$

($\|\vec{v}_1\| = \|\vec{v}_2\| = \|\vec{v}_3\| = 1$ and $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3 = 0$)

$$\Rightarrow A^{-1} = A^T = \frac{1}{7}\begin{bmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ 3 & 6 & 2 \end{bmatrix}$$

Prop Every symmetric matrix A has a diagonalization

$$A = PDP^T$$

where the columns of P form an orthonormal eigenbasis for A.

Note We have $P^{-1} = P^T$ as the columns of P are given by an orthonormal eigenbasis for A.

Ex Find the solution of the dynamical system $\vec{x}_{n+1} = A\vec{x}_n$ with

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \text{ and } \vec{x}_0 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Sol A is symmetric $\Rightarrow$ A is diagonalizable

$$P_A(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 2 & 0 \\ 2 & 1-\lambda & 2 \\ 0 & 2 & 2-\lambda \end{bmatrix}$$

$$= -\lambda \det\begin{bmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} - 2\det\begin{bmatrix} 2 & 0 \\ 2 & 2-\lambda \end{bmatrix}$$

$$= -\lambda\left[(1-\lambda)(2-\lambda) - 4\right] - 2\cdot 2(2-\lambda)$$

$$= -\lambda(\lambda^2 - 3\lambda - 2) - 4(2-\lambda)$$

$$= -\lambda^3 + 3\lambda^2 + 6\lambda - 8 = -(2+\lambda)(1-\lambda)(4-\lambda)$$

$\Rightarrow$ A has eigenvalues $\lambda = -2, 1, 4$

$$A + 2I = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix} \Rightarrow \text{RREF}(A+2I) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A+2I)\vec{x} = \vec{0} \Rightarrow \begin{cases} x_1 - 2x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2x_3 \\ x_2 = -2x_3 \end{cases} \xRightarrow[x_3=t]{} \vec{x} = t\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$A - I = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow \text{RREF}(A-I) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A-I)\vec{x} = \vec{0} \Rightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 + 0.5x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = -0.5x_3 \end{cases} \xRightarrow[x_3=t]{} \vec{x} = \frac{t}{2}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

$$A-4I=\begin{bmatrix}-4&2&0\\2&-3&2\\0&2&-2\end{bmatrix}\Rightarrow \mathrm{RREF}(A-4I)=\begin{bmatrix}1&0&-0.5\\0&1&-1\\0&0&0\end{bmatrix}$$

$$(A-4I)\vec{x}=\vec{0}\Rightarrow\left\{\begin{array}{l}x_1 \quad -0.5x_3=0\\ \quad x_2-\quad x_3=0\end{array}\right.\Rightarrow\left\{\begin{array}{l}x_1=0.5x_3\\x_2=\quad x_3\end{array}\right.\xRightarrow[x_3=t]{}\vec{x}=\frac{t}{2}\begin{bmatrix}1\\2\\2\end{bmatrix}$$

Hence an eigenbasis for A is given by

$$\vec{v}_1=\begin{bmatrix}2\\-2\\1\end{bmatrix},\ \vec{v}_2=\begin{bmatrix}-2\\-1\\2\end{bmatrix},\ \vec{v}_3=\begin{bmatrix}1\\2\\2\end{bmatrix}$$

Since A is symmetric, the eigenvectors $\vec{v}_1,\vec{v}_2,\vec{v}_3$ are orthogonal

($\vec{v}_1,\vec{v}_2,\vec{v}_3$ correspond to distinct eigenvalues)

$\Rightarrow$ An orthonormal eigenbasis for A is given by

$$\frac{\vec{v}_1}{\|\vec{v}_1\|}=\frac{1}{3}\begin{bmatrix}2\\-2\\1\end{bmatrix},\ \frac{\vec{v}_2}{\|\vec{v}_2\|}=\frac{1}{3}\begin{bmatrix}-2\\-1\\2\end{bmatrix},\ \frac{\vec{v}_3}{\|\vec{v}_3\|}=\frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}.$$

$\Rightarrow$ A has a diagonalization $A=PDP^T$ with

$$P=\frac{1}{3}\begin{bmatrix}2&-2&1\\-2&-1&2\\1&2&2\end{bmatrix}\quad \text{and}\quad D=\begin{bmatrix}-2&0&0\\0&1&0\\0&0&4\end{bmatrix}$$

$$\begin{array}{ccc}\frac{\vec{v}_1}{\|\vec{v}_1\|}&\frac{\vec{v}_2}{\|\vec{v}_2\|}&\frac{\vec{v}_3}{\|\vec{v}_3\|}\end{array}$$

eigenvalues for $\vec{v}_1,\vec{v}_2,\vec{v}_3$ (in order!)

$$\Rightarrow A^n=PD^nP^T=\frac{1}{3}\begin{bmatrix}2&-2&1\\-2&-1&2\\1&2&2\end{bmatrix}\begin{bmatrix}(-2)^n&0&0\\0&1^n&0\\0&0&4^n\end{bmatrix}\frac{1}{3}\begin{bmatrix}2&-2&1\\-2&-1&2\\1&2&2\end{bmatrix}$$

$$\Rightarrow \vec{x}_n=A^n\vec{x}_0=\underbrace{\frac{1}{9}\begin{bmatrix}2&-2&1\\-2&-1&2\\1&2&2\end{bmatrix}\begin{bmatrix}(-2)^n&0&0\\0&1&0\\0&0&4^n\end{bmatrix}\begin{bmatrix}2&-2&1\\-2&-1&2\\1&2&2\end{bmatrix}}_{A^n}\underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{\vec{x}_0}=\boxed{\frac{1}{9}\begin{bmatrix}-2\cdot(-2)^n-8+4^n\\2\cdot(-2)^n-4+2\cdot4^n\\-(-2)^n+8+4^n\end{bmatrix}}$$